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Dynamical and spatial disorder in an intermittent search process

M Moreau, O Bénichou, C Loverdo and R Voituriez

Laboratoire de Physique Théorique de la Matière Condensée, Université Pierre et Marie Curie, UMR CNRS 7600, 75252 Paris Cedex 05, France

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Abstract
We consider a one-dimensional model of an intermittent search process in a medium exhibiting frozen disorder. A tracer, searching for Poisson-distributed targets, alternates diffusive and ballistic motions, but can only find a target when diffusing. Preliminary theoretical results [1] are now confirmed, completed and extended, and their derivations are presented for the first time. We study the mean search time $\langle T \rangle$ according to the laws of the searcher waiting times in the diffusive and ballistic regimes. In particular, we obtain a lower bound of $\langle T \rangle$, which in certain circumstances is also an approximation and is valid for a very broad class of waiting time distributions. Explicit results and other approximations are presented in the case of exponential waiting times, and we study the optimization of $\langle T \rangle$, depending on the mean durations of the diffusive and ballistic phases. Theoretical formulae are supported by numerical simulations. We show that the intermittent behaviour can allow one to minimize the search time in comparison with the purely diffusive behaviour, and that it is possible, by an adequate choice of the parameters, to increase very significantly the efficiency of the search.

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1. Introduction

Search processes have been intensively studied over many years [2–9] for practical applications, such as research of mines after the 2nd world war or rescue operations, or in natural sciences, like behavioural biology, physics or chemistry, the main problem being that a 'searcher' has to find a 'target' as rapidly as possible. Examples can be found in all fields and at all scales, from predators searching for prey [7, 8, 10–13] to mobile molecules, or tracers, able to be captured by specific receptors [3–6, 14–19]. Recently, a new burst of interest occurred when stochastic processes were used to investigate the case when the searcher has no indication on the location of the target and has to move blindly till it reaches it [20–23]. In general, the

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scanning of the search region is slow, whereas the searcher can move much more rapidly but with lower or non-existent detection abilities. In this case, it was shown analytically [22–25] that in certain conditions, the searcher may profitably alternate slow scanning phases with fast displacements, which do not allow for detection but lead rapidly to unexplored regions. This intermittent behaviour is in fact observed in many cases, for instance for foraging animals [7, 8], and simple models yield results which are confirmed by observations. Since then, many articles have been published on these points (see for instance [26–36]) in order to extend the theory. Most of the models essentially concern one-dimensional motions: in fact, besides being simpler for calculations, they have especially interesting properties and are relevant for certain real-life situations. They present, however, a serious limitation by using either a unique target in a finite space interval or, equivalently, an infinite number of regularly spaced targets. In fact, in many cases the searcher is looking for any target, among many identical ones, as do for instance most predators, or reactive molecules in quest of a reactant, and the targets are not regularly ordered. As a typical case, one can consider the case of a Poisson distribution of targets, which is in some sense the opposite of a completely ordered medium with regularly distributed targets. In a short, preliminary paper [1], we gave the first results for a one-dimensional Poisson distribution of targets, announcing that the search time can be minimized as in the case of an ordered distribution, by conveniently choosing the mean waiting times in the slow and fast phases, but with different laws. In the disordered case, however, the analytical results can be deduced not from exact solutions of the equations, but from rather intricate approximations which were not described in detail up to now. The purpose of this paper is to study the dynamical disorder in a one-dimensional intermittent search process which generalizes the model presented in [1], to describe the corresponding approximations completely, and to prove and discuss the results.

First, we present the model and the equations. In the next section, we discuss two simple limit cases and give the corresponding approximate results. Then, we present the intermediary approximation, and give the corresponding, rather lengthy calculations (some of the details being left for appendices). Eventually, we discuss the results and compare them to numerical simulations.

2. Model and equations

2.1. Model

We consider a point **P**, modelling a 'searcher' which can be a molecule or a group of molecules, or on the macroscopic scale, a person, an animal, or any device able to detect some special sites called 'targets'. **P** moves on an infinite straight line containing a countable set of identical targets, located at some fixed points $A_1, A_2, A_k, \ldots$. All targets are supposed to be equivalent for the searcher. The position of **P** is determined by its coordinate $x(t)$ on an axis Ox along the straight line. The position of target A_k on Ox is denoted a_k, and the set $\eta \equiv \{a_k\}$ is the targets repartition. The system evolves according to the following rules:

- *Searcher dynamics*. **P** can obey two distinct dynamic regimes, denoted by 1 and 2.
  - In regime 1, **P** performs a Brownian motion with diffusion coefficient D. **P** finds a target as soon as it reaches its position A during this regime.
  - In regime 2, **P** performs a ballistic motion with the constant velocity v ($v > 0$). **P** cannot find any target during this regime.
- *Waiting times*. Each regime i ($i = 1$ or 2) has a finite, stochastic duration T_i, independent of other variables: its survival probability is $\phi_i(t) \equiv P(T_i > t)$, and its probability density is $\varphi_i(t) = -\mathrm{d}\phi_i(t)/\mathrm{d}t$. Unless differently specified, we will assume that T_i has a finite

mean value $\langle T_i \rangle = \tau_i$. In similar studies, it is often assumed that T_i follows an exponential law: $\phi_i(t) = \exp(-t/\tau_i)$, in which case the system is Markovian. However, depending on the problem to be modelled, other laws can be more relevant, leading to a non-Markov evolution. This is the case, for instance, if T_i is deterministic: $T_i = \tau_i$. For the moment, however, it is not necessary to specify the law of T_i.

At the end of a regime, the system switches instantaneously to the other regime.

- *Targets distribution.* We assume that the targets obey a Poisson distribution with a uniform density ν, so that the probability that an interval of length l contains no target is $P(l) = \mathrm{e}^{-\nu l}$.

In many search processes it is important to study the first time T when some target is discovered, which coincides with the first passage time of **P** on a target in regime 1. We now estimate its mean value, averaged on the dynamical disorder due to the stochastic motions of P, and on the disorder in the distribution of the targets.

2.2. Equations

The stochastic state of the complete system at time t is described by the position x of **P**, its dynamical regime i, and the repartition $\eta \equiv \{a_h\}$ of the targets. Both x and i change with time, whereas η is frozen. We assume that at time $t = 0$, regime 1 is initiated and **P** starts from point x_0. Let us denote $T(x_0, i = 1, \eta)$ the time of the first discovery of some target, starting from x_0 in regime 1 at time 0, with repartition η of targets. We define the 'survival probability' $S(t|x_0)$ that P, starting from x_0 in regime 1 at time 0, has found no target at time t, averaged on the distribution η of targets, or with obvious notations

$$\begin{aligned} S(t|x_0) &= \langle S(t\,|x_0, 1, \eta)\rangle_\eta \\ &\equiv \langle \text{Proba(no target has been found before } t\,|x_0, i = 1, \eta)\rangle_\eta. \end{aligned}$$

Here $S(t|x_0, i = 1, \eta)$ is the probability of the set of trajectories which do not reach any target during regime 1 between times 0 and t: we call such a trajectory 'non-reactive'. The average $\langle\ \rangle_\eta$ is taken on the Poisson distribution of η.

Similarly, the conditional density probability $P(t|x_0) = -\,\partial S(t|x_0)/\partial t$ that the first target is discovered at time t, knowing that **P** starts from x_0 in regime 1 at time 0, is averaged on η.

Let us now consider a particular set Γ of trajectories of the stochastic process *in the absence of targets*: they start from x_0 in regime 1 and reach x in regime 1 between t and $t + \mathrm{d}t$, and undergo $2n$ changes of regime between times 0 and t, at the successive times $t_1 < t_2 < \cdots t_{2n}$, with n being some non-negative integer. We denote x_i the position of **P** at time t_i. Thus, during the interval $[t_{2i}, t_{2i+1}[$, **P** obeys the diffusive regime 1. If $i = 0, 1, \ldots, n-1$, we denote b_{2i+1} and c_{2i+1}, respectively, the lowest and the largest position of **P** during $[t_{2i}, t_{2i+1}[$:

$$b_{2i+1} = \inf x(t'), \qquad c_{2i+1} = \sup x(t') \qquad \text{for} \quad t'[t_{2i}, t_{2i+1}[\tag{1}$$

$$b_{2n+1} = \inf x(t'), \qquad c_{2i+1} = \sup x(t') \qquad \text{for} \quad t'[t_{2n}, t[. \tag{2}$$

Let $\mathrm{d}x_i$, $\mathrm{d}b_{2i+1}$, $\mathrm{d}c_{2i+1}, \ldots$ be the differentials of these respective variables. We denote

$$\begin{aligned} &P_{2n}(t, x, b_{2n}, c_{2n}; t_{2n}, x_{2n}; t_{2n-1}, x_{2n-1}, b_{2n-1}, c_{2n-1}; \ldots t_1, x_1, b_1, c_1|x_0)\,\mathrm{d}t \\ &\qquad \mathrm{d}x_{2n+1}\,\mathrm{d}b_{2n+1}\,\mathrm{d}c_{2n+1}\,\mathrm{d}t_{2n}\,\mathrm{d}x_{2n}\ldots\mathrm{d}t_1\,\mathrm{d}x_1\,\mathrm{d}b_1\,\mathrm{d}c_1 \end{aligned}$$

the probability density of this set of trajectories. Since the evolution during regimes 1 and 2 are Markov processes, and the changes of regimes are independent events, we can write with the previous notations

$$
\begin{aligned}
P_{2n}(\Gamma) \equiv\ & P_{2n}(t, x, b_{2n}, c_{2n}; t_{2n}, x_{2n}; t_{2n-1}, x_{2n-1}, b_{2n-1}, c_{2n-1}; \ldots t_1, x_1, b_1, c_1|x_0) \\
=\ & \phi_1(t - t_{2n}) p_1(t - t_{2n}, x, b_{2n+1}, c_{2n+1}|x_{2n-2}) \varphi_2(t_{2n} - t_{2n-1}) \\
& \times\ p_2(t_{2n} - t_{2n-1}, x_{2n}|x_{2n-1}) \cdot \varphi_1(t_{2n-1} - t_{2n-2}) \\
& \times\ p_1(t_{2n-1} - t_{2n-2}, x_{2n-1}, b_{2n-1}, c_{2n-1}|x_{2n-2}) \ldots \varphi_1(t_1) p_1(t_1, x_1, b_1, c_1|x_0),
\end{aligned} \tag{3}
$$

where we used the following definitions:

- $\phi_i(t)$ is the survival probability of regime i after time t, $\varphi_i(t)$ is its corresponding time-life probability density, as specified previously.
- $p_1(t_1, x_1, b_1, c_1|x_0)$ is the conditional probability density at time t_1 to be in x_1, with upper and lower bounds b_1 and c_1 for $x(t')$, $t' \in [0, t_1[$, knowing that **P** starts from x_0 at time 0 and that regime 1 is maintained during $[0, t_1[$.
- $p_2(t_2 - t_1, x_2 \mid x_1)$ is the propagator from x_1 to x_2 in time $t_2 - t_1$ during regime 2.

The trajectories belonging to the set Γ are non-reactive if and only if there is no target in all space intervals $I_{2k+1} \equiv [b_{2k+1}, c_{2+1}]$ for $k = 0, 1, n$. Such an event is measurable with respect to the set Γ, and its conditional probability density, averaged on the Poisson distribution of targets, is denoted $F(\{I_{2k+1}\})$: it only depends on the intervals I_{2k+1}. It is the conditional survival probability of **P** at time t, knowing the set of trajectories Γ, averaged on the repartition η of targets. Thus, the survival probability of **P** for trajectories with $2n$ changes of regimes (including the probability to have $2n$ changes) is

$$
S_{2n}(t|x_0) = \int \prod_{0 \leqslant k \leqslant n} \mathrm{d}t_{2k}\, \mathrm{d}t_{2k+1}\, \mathrm{d}x_{2k}\, \mathrm{d}x_{2k+1}\, \mathrm{d}b_{2k+1}\, \mathrm{d}c_{2k+1}\, \mathrm{d}x_{2k}\, \mathrm{d}_{2k+1} P_0(\Gamma) F(\{I_{2k+1}\}) \tag{4}
$$

the integration domains being specified by writing $t_{2n+1} \equiv t$ and $x_{2n+1} \equiv x$ and

$$
\begin{aligned}
& t_1 < t_2 < \cdots t_{2n} < t_{2n+1}, \ldots; \qquad -\infty < x_k < +\infty; \\
& \qquad -\infty < b_{2k+1}, < x_{2k+1}, \qquad \text{and} \qquad x_{2k+1} < c_{2k+1} < +\infty.
\end{aligned}
$$

Formula (5) can be simplified by using the Laplace transforms of function $F(t)$:

$$
\tilde{F}(s) = \int_{t \geqslant 0} F(t)\, \mathrm{e}^{-\mathrm{st}}
$$

and noticing that $p_1(t, x, b, c|t_0, x_0)$ and $p_2(t, x|x_0)$, being defined for the free evolutions of regimes 1 and 2, respectively, are invariant by translation, so that, writing $\tau = t - t_0$, $y = x - x_0$, $\beta = b - x_0$, $\chi = c - x_0$, we can write for instance $p_1(t, x, b, c|t_0, x_0) \equiv p_1(\tau, y, \beta, \chi)$. Then

$$
\begin{aligned}
\tilde{S}_{2n}(s|x_0) = \int & \mathrm{d}y\, \mathrm{d}\beta_{2n+1}\, \mathrm{d}\chi_{2n+1} Q_1(s, y, \beta_{2n+1}, \chi_{2n+1}) \\
& \cdot \left[\int \prod_{1 \leqslant k \leqslant n} \mathrm{d}y_{2k}\, \mathrm{d}y_{2k-1}\, \mathrm{d}\beta_{2k-1}\, \mathrm{d}\chi_{2k-1} q_2(s, y_{2k}) q_1(s, y_{2k-1}, \beta_{2k-1}, \chi_{2k-1}) \right] \\
& \times F(I_1, \ldots I_{2n+1}),
\end{aligned} \tag{5}
$$

where $-\infty < \beta_{2k+1} < 0$, $0 < \chi_{2k+1} < +\infty$, $\beta_{2k+1} < y_{2k+1} < \chi_{2k+1}$, $0 < y_{2k} < \infty$, and

- $Q_1(s, y, \beta, \chi)$ is the Laplace transform of $\phi_1(t)\, p_1(t, y, \beta, \chi)$ (which is $\tilde{p}_1(s + \lambda_1, y, \beta, \chi)$ if the duration T_1 of regime 1 is an exponential stochastic variable: $\phi_1(t) = \exp(-\lambda_1 t)$).
- $q_1(s, x, b, c|x_0)$ is the Laplace transform of $\varphi_1(t)$ $p_1(t, x, b, c|x_0)$ (which is $\lambda_1 \tilde{p}_1(s + \lambda_1, y, \beta, \chi)$ if the duration T_1 of regime 1 is an exponential stochastic variable).

- $Q_2(s, x)$ is the Laplace transform of $\phi_2(t)\, p_2(t, x)$ (which is $\tilde{p}_2(s + \lambda_2, x)$ if the duration T_2 of regime 2 is an exponential stochastic variable: $\phi_2(t) = \exp(-\lambda_2 t)$).
- $q_2(s, x|x_0)$ is the Laplace transform of $\varphi_2(t)\, p_2(t, x, |x_0)$ (which is $\lambda_2\, \tilde{p}_2(s + \lambda_2, y, \beta, \chi)$ if the duration T_2 of regime 2 is an exponential stochastic variable).

If, now, we consider trajectories with $2n + 1$ changes of regime, the last regime is the ballistic phase 2 and it is easily found that the corresponding survival probability is

$$\begin{aligned}\tilde{S}_{2n+1}(s|x_0) = \tilde{\phi}_2(s) \int \mathrm{d}y_{2n+1}\, \mathrm{d}\beta_{2n+1}\, \mathrm{d}\chi_{2n+1} q_1(s, y_{2n+1}, \beta_{2n+1}\chi_{2n+1}) \\ \cdot \left[\int \prod_{1\leqslant k\leqslant n} \mathrm{d}y_{2k}\, \mathrm{d}y_{2k-1}\, \mathrm{d}\beta_{2k-1}\, \mathrm{d}\chi_{2k-1} q_2(s, y_{2k}) q_1(s, y_{2k-1}, \beta_{2k-1}, \chi_{2k-1}) \right] \\ \times F(I_1, \ldots I_{2n+1}),\end{aligned} \tag{6}$$

where $\tilde{\phi}_2(s)$ is the Laplace transform of $\phi_2(t)$. The Laplace transform of $S(t|x_0)$ probability is

$$\tilde{S}(s|x_0) = \sum_{n\geqslant 0} [\tilde{S}_{2n}(s|x_0) + \tilde{S}_{2n+1}(s|x_0)]. \tag{7}$$

In general, it is quite difficult to compute expressions (5) and (6), since all variables β_{2k+1} and χ_{2k+1} are implied in $F(\{I_{2k-1}\})$, so that the integrals of equations (5) and (6) are coupled. In two special cases, however, the probability $F(\{I_k\})$ can be expressed very simply and reasonable approximations allow one to compute expression (7) explicitly:

(i) if all intervals I_k are disjoint, the properties of the Poisson distribution imply

$$F(\{I_k\}) = \prod_{i=0,\ldots,n} \exp[-\nu(\chi_{2i+1} - \beta_{2i+1})], \tag{8}$$

(ii) if in contrast the intervals I_i strongly overlap, so that $\cup_{i=0,\ldots n} I_{2i+1}$ is some interval (b, c), we have simply

$$F(\{I_k\}) = \exp[-\nu(c - b)]. \tag{9}$$

Clearly these cases are particular and do not cover all possible trajectories. Nevertheless, we will see that they lead to lower bounds for the survival probability $S(t|x_0)$. Furthermore, these lower bounds are also approximate values of $S(t|x_0)$ in some conditions which we now study. In fact, in all cases, we can use the inequality

$$F(\{I_k\}) \geqslant \prod_{i=0,\ldots,n} \exp[-\nu(\chi_{2i+1} - \beta_{2i+1})], \tag{10}$$

which changes into equality if all intervals I_k are disjoint, due to the properties of the Poisson distribution. If the I_k are not disjoint, inequality (10) holds because the probability of common intersections is factorized several times in the right-hand side of (10), but only once in the left-hand side. This lower bound can be computed and yields an approximate value in case (i). On the other hand, another lower bound of the survival probability is obtained by using the inequality

$$F(\{I_k\}) \geqslant \exp[-\nu(c - b)] \qquad \text{with} \quad c = \inf_{0\leqslant i\leqslant n} c_{2i+1} \quad \text{and} \quad b = \sup_{0\leqslant i\leqslant n} b_{2i+1} \tag{11}$$

since $\cup_k I_k \subset (b, c)$. These approximations are considered in section 3.

3. Bounds and approximations of the survival probability

3.1. Lower bound of the survival probability

When replacing $F(\{I_k\})$ by the lower bound (10) in equations (5) and (6), the integrals separate and give for instance

$$\tilde{S}_{2n}(s|x_0) \geqslant \int \mathrm{d}y\,\mathrm{d}\beta_{2n+1}\,\mathrm{d}\chi_{2n+1}\,Q_1(s,y,\beta_{2n+1},\chi_{2n+1})\exp(-\nu(\chi_{2n+1}-\beta_{2n+1}))$$
$$\cdot\int\prod_{1\leqslant k\leqslant n}\mathrm{d}y_{2k}\,\mathrm{d}y_{2k-1}\,\mathrm{d}\beta_{2k-1}\,\mathrm{d}\chi_{2k-1}$$
$$\times\, q_2(s,y_{2k})q_1(s,y_{2k-1},\beta_{2k-1},\chi_{2k-1})\exp(-\nu(\chi_{2k-1}-\beta_{2k-1}) \tag{12}$$

with $-\infty < \beta_{2k-1} < 0$, $0 < \chi_{2k-1} < +\infty$, $\beta_{2k-1} < y_{2k-1} < \chi_{2k-1}$ and $0 < y_{2k} < +\infty$. Thus we obtain

$$\tilde{S}_{2n}(s|x_0) \geqslant K(s,\nu)[q_2(s)k(s,\nu)]^n; \qquad \tilde{S}_{2n+1}(s|x_0) \geqslant Q_2(s)k(s,\nu)[q_2(s)k(s,\nu)]^n \tag{13}$$

with

$$q_2(s) \equiv \int \mathrm{d}y\, q_2(s,y) = \tilde{\varphi}_2(s); \qquad Q_2(s) \equiv \int \mathrm{d}y\, Q_2(s,y) = \tilde{\Phi}_2(s)$$
$$k(s,\nu) = \int_{-\infty<\beta<0}\mathrm{d}\beta\int_{0<x<\infty}\mathrm{d}\chi\int_{-\beta<y<\chi}\mathrm{d}y\; q_1(s,y,\beta,\chi)\exp(-\nu(\chi-\beta)) \tag{14}$$
$$K(s,\nu) = \int_{-\infty<\beta<0}\mathrm{d}\beta\int_{0<x<\infty}\mathrm{d}\chi\int_{-\beta<y<\chi}\mathrm{d}y\; Q_1(s,y,\beta,\chi)\exp(-\nu(\chi-\beta))$$

and eventually, by (8)

$$\tilde{S}(s\,|x_0) \geqslant \frac{K(s,\nu)+Q_2(s)k(s,\nu)}{1-q_2(s)k(s,\nu)} \equiv \underline{\tilde{S}}(s). \tag{15}$$

The probability of never finding any target is $\underline{S}(\infty) = \lim_{s\to 0} s\underline{\tilde{S}}(s)$. We note that $K(s,\nu)$ is the Laplace transform of the probability that at time t the phase 1 has not finished and that the searcher has found no target, which is obviously smaller than the survival probability $\phi_1(t)$ of phase 1 at t. Thus $sK(s,\nu) < s\,\tilde{\phi}_1(s) \to 0$ if $\phi_1(\infty) = 0$, which we suppose. Furthermore, $k(s,\nu)$ is the Laplace transform of the probability density that phase 1 finishes at t and that the searcher has found no target at this moment, which is clearly less than $\varphi_1(t)$, so that $k(s,\nu) < 1$. On the other hand, if $s\to 0$, $s\tilde{Q}_2(s) = 1 - q_2(s) \to 0$ if we also suppose that $\phi_2(\infty) = 0$, and we have $\tilde{Q}_2(0) = \tilde{\phi}_2(0) = \tau_2$, the average duration of regime 2, which we supposed to be finite. Then it is seen from (15) that

$$\underline{S}(\infty) = \lim_{s\to 0} s\underline{\tilde{S}}(s) = 0.$$

This is only a lower bound for $S(\infty|x_0)$, but we will now admit, and show later, that $S(\infty|x_0)$ also vanishes, so that the searcher has probability 1 to find some target at a finite time.

Next, our main interest is to compute the average search time, starting from x_0 in regime 1, which is

$$\langle T\rangle = \int_{0<t}\mathrm{d}t\, t\,P(t|x_0) = \int_{0<t}\mathrm{d}t\; S(t|x_0) = \tilde{S}(s=0) \geqslant \underline{\tilde{S}}(s=0).$$

Thus by (15), we obtain our first main result

$$\langle T\rangle \geqslant \frac{K(0,\nu)+\tau_2 k(0,\nu)}{1-k(0,\nu)}. \tag{16}$$

It should be noted that this lower bound of $\langle T\rangle$ does not depend on the precise law for T_2, but only on the average duration τ_2, provided it is finite.

3.2. Approximation of large ballistic displacements

We now assume that there is a time $\eta_2 < \tau_2$ such that the probability that $T_2 < \eta_2$ is very low and can be neglected. This is clearly the case if the density of T_2 has a sharp maximum around its average τ_2, and in particular if the lifetime T_2 of regime 2 is deterministic. It should be remarked that if T_2 follows an exponential law, whose density is maximum at 0, this cut-off neglects very probable trajectories and should be discussed carefully. Adopting, nevertheless, this hypothesis, we can consider that the displacement during a ballistic phase is at least $v\eta_2$.

On the other hand, we also assume that there is a time $\theta_1 > \tau_1$ such that the probability of $T_1 > \theta_1$ is very low and can be neglected. Furthermore, we assume that the typical displacement γ_1 during regime 1, if $T_1 < \theta_1$, is much smaller than $v\eta_2$: $\gamma_1 \sim (2D\theta_1)^{1/2} \ll v\eta_2$. Then we can neglect the diffusive trajectories including displacements larger than $v\eta_2$. If the durations of both regimes i are sharply peaked on their typical values τ_i, this situation occurs when $(2D\tau_1)^{1/2} \ll v\tau_2$, but the latter condition can be insufficient if the durations are exponential, because of the large fluctuations of this law from its mean value. This case will be discussed specifically later.

With the previous approximations, the intervals $\{I_{2k-1}\}$ defined previously are disjoint, equation (8) applies, and the lower bounds computed previously yield the approximate values

$$\tilde{S}(s|x_0) \sim \frac{K(s,\nu) + Q_2(s)k(s,\nu)}{1 - q_2(s)k(s,\nu)} \equiv \underline{\tilde{S}}(s) \tag{17}$$

and for the average search time

$$\langle T\rangle \sim \frac{K(0,\nu) + \tau_2 k(0,\nu)}{1 - k(0,\nu)}. \tag{18}$$

Once more, we emphasize that this value only depends on the average duration τ_2. Furthermore, it can be shown from (17) that, if regime 1 has a finite average waiting time τ_1, whereas the waiting time T_2 of regime 2 follows a heavy-tailed law [19, 34] with exponent $\alpha \in\,]0, 1[$, i.e. if $\Phi_2(t) \equiv P(T_2 > t) \propto t^{-\alpha}$ if $t \to \infty$, the average duration of T_2 is infinite and the overall search time $T(x_0)$ asymptotically follows the same law up to a renormalization factor. More precisely, if $t \to \infty$

$$P(T(x_0) > t) \equiv S(t|x_0) \sim P(\gamma T_2 > t) \equiv \Phi_2\left(\frac{t}{\gamma}\right) \qquad \text{with} \quad \gamma = \left(\frac{k(0,\nu)}{1 - k(0,\nu)}\right)^{1/\alpha}$$

or

$$S(t|x_0) \sim \frac{k(0,\nu)}{1 - k(0,\nu)} \Phi_2\left(\frac{t}{\gamma}\right) \qquad \text{if} \quad t \to \infty. \tag{19}$$

These results may be compared with those obtained in different cases in [19, 34]. Here, we remark that if T_2 follows a heavy-tailed law with exponent $\alpha \in\,]0, 1[$, which has an infinite average, the overall search time is infinite and intermittence is obviously unfavourable to the search, whereas if T_2, as in [34], follows a heavy-tailed law with exponent $\alpha \in\,]1, 2[$, which has a finite average, formula (18) still applies and the intermittence may allow one to decrease the search time. In order to study this possibility precisely, we need to specify the waiting time distributions.

3.3. Exponential waiting time in diffusive regime

We now return to the case when regime 2 has a finite average waiting time τ_2. In order to obtain more explicit expressions, we assume that T_1 follows an exponential law: $P(T_1 > t) = \exp(-\lambda_1 t)$, with $\lambda_1 = 1/\tau_1$. Then

$$\begin{aligned} q_1(s, y, \beta, \chi) &= \lambda_1 \tilde{p}_1(s + \lambda_1, y, \beta, \chi) \\ Q_1(s, y, \beta, \chi) &= \tilde{p}_1(s + \lambda_1, y, \beta, \chi). \end{aligned} \tag{20}$$

We see that $K(0, \nu) = \tau_1\, k(0, \nu)$ and (17) reads

$$\langle T\rangle = \frac{(\tau_1 + \tau_2)\, k(0, \nu)}{1 - k(0, \nu)}, \tag{21}$$

where $k(0, \nu)$ depends on τ_1, but not on τ_2, so that the average time $\langle T\rangle$, given by (21), is always an increasing function of τ_2 for fixed τ_1. This is no surprise since in the present conditions the searcher does not explore completely the regions where it moves, due to the long ballistic phases, and increasing them too much will still make the research less efficient. On the other hand, $k(0, \nu)$ can be a non-monotonic function of τ_1, which may lead to non-trivial behaviour of $\langle T\rangle$ with respect to τ_1.

The probability density $p_1(\tau, y, \beta, \chi)$ to be in y at time τ after undergoing the maximal negative and positive deviations β and χ, starting from the origin, is given by

$$p_1(t, y, \beta, \chi) = -(\partial^2/\partial\beta\partial\chi)\, p_1(t, y|\beta, \chi), \tag{22}$$

where $p_1(t, y|\beta, \chi)$ is the probability density to be at y at time t in the diffusive regime, starting from the origin and knowing that β and χ are absorbing points. This quantity is well known, but we will only use the Laplace transform of

$$p_1(t|\beta, \chi) = \int_{\beta<y<\chi} \mathrm{d}y\; p_1(t, y|\beta, \chi),$$

which can easily be computed directly [37, 38]

$$\tilde{p}_1(s\,|\beta, \chi\,) = \frac{1}{s}\left[1 - \frac{\mathrm{sh}(\sqrt{s/D}\chi) - \mathrm{sh}(\sqrt{s/D}\beta)}{\mathrm{sh}(\sqrt{s/D}(\chi - \beta))}\right] = \frac{1}{s}\left[1 - \frac{\mathrm{ch}(\sqrt{s/D}(\chi + \beta)/2)}{\mathrm{ch}(\sqrt{s/D}(\chi - \beta)/2)}\right]. \tag{23}$$

Then we find according to the previous notations,

$$K(s, \nu) = \tau_1 k(s, \nu) = \int_{-\infty<\beta<0} \mathrm{d}\beta \int_{0<\chi<\infty} \mathrm{d}\chi\, \tilde{p}_1(s + \lambda_1|\beta, \chi)\, \nu^2 \exp(-\nu(\chi - \beta)). \tag{24}$$

Writing $\kappa = \chi - \beta$, $\alpha = (D\tau_1)^{-1/2}$ and $\mu = \alpha/(2\nu)$, it is found from (24)

$$k(0, \nu) = 1 - 2\,\alpha^{-1}\nu^2 \int_{0<x<\infty} \mathrm{d}\kappa\, \mathrm{e}^{-\nu\kappa} \tanh(\alpha\kappa/2) = 1 - \mu^{-1} \int_{0<x<\infty} \mathrm{d}x\, \mathrm{e}^{-x} \tanh(\mu x). \tag{25}$$

We remark that in the case of a purely diffusive motion, without changes in the regime, the average search time is obtained from (24) when $\mu_1 \to 0$ (or $\tau_1 \to \infty$: the initial diffusive regime is maintained indefinitely). Then we have $\mu \to 0$ and $\tanh(\mu x) = \mu x - (\mu x)^3/3 + \cdots$, which gives

$$k(0, \nu) \sim 2\mu^2 = 2(2\nu)^{-2}(D\tau_1)^{-1}$$

so that the average search time in the purely differential regime is, as expected, proportional to the square of the mean distance $L = 1/\nu$ between the two targets

$$\langle T\rangle_{\mathrm{diff}} = K(0, \nu) = \lim_{\tau_1\to\infty} \tau_1 k(0, \nu) = \tfrac{1}{2}\nu^{-2}D^{-1} = L^2/(2D). \tag{26}$$

Returning to equation (25) in the case of a small target density ν, such that $\nu(D\tau_1)^{1/2} \ll 1$, or $\mu \gg 1$, we have $\tanh(\mu x) \sim 1$ so that

$$k(0, \nu) \sim 1 - 2\nu(Dt_1)^{1/2} \tag{27}$$

and since $\nu(D\tau_1)^{1/2} \ll 1$

$$\langle T\rangle \sim (\tau_1 + \tau_2)\frac{1}{2\nu(D\tau_1)^{1/2}}. \tag{28}$$

According to this formula, in order to make $\langle T\rangle$ as small as possible for a given value of τ_1, one should chose τ_2 as small as possible, but it should satisfy the condition $v\tau_2 \gg (D\tau_1)^{1/2}$ for the present approximation to be valid. It will be seen in section 4.1 that, in these conditions, $\langle T\rangle$ can be much smaller than $\langle T\rangle_{\text{diff}}$, given by (26), provided that

$$\nu \ll \frac{v}{D} \quad \text{or equivalently} \quad \frac{L}{v} \ll \frac{L^2}{D}, \tag{29}$$

which is satisfied if the density is low enough. Thus, the time needed to cover the average distance between two targets should be much shorter in the ballistic regime than in the diffusing regime: this quite understandable condition allows reducing significantly the mean search time thanks to intermittence.

It should be pointed out that the mean search time is proportional to the mean distance $L = \nu^{-1}$ between two neighbouring targets, whereas $\langle T\rangle$ scales as L^2 if the searcher obeys a purely diffusive motion. Thus, even if the optimal choices of τ_1 and τ_2 are not realized, the intermittent strategy is justified in the case of a small target density, provided that the duration of the ballistic phases satisfies the previous inequalities.

3.4. Approximation of small ballistic displacements

We now assume that the average displacement $v\tau_2$ during a ballistic phase is significantly smaller than the characteristic displacement $(2D\tau_1)^{1/2}$ during a diffusive phase. Thus, with a high probability the intervals I_{2k+1} scanned during the diffusive phases overlap and satisfy the condition

$$\cup_{i=0,\dots,n} I_{2i+1} = (c-b) \quad \text{with} \quad c = \inf_{0\leqslant i\leqslant n} c_{2i+1} \quad \text{and} \quad b = \sup_{0\leqslant i\leqslant n} b_{2i-1}$$

so that in formulae (6) and (7)

$$F(\{I_k\}) = \exp[-\nu(c-b)]. \tag{30}$$

It is not easy to characterize the values b and c but, because of the systematic positive drift due to the ballistic phases, we can consider that with a high probability

$b \sim b_1$, the lower bound of the searcher trajectory during the first diffusive phase

$c \sim c_{2n+1}$, the upper bound of the searcher trajectory during the last diffusive phase

so that (30) is replaced by

$$F(\{I_k\}) \sim \exp\left[-\nu(c_{2n+1} - b_1)\right]. \tag{30$'$}$$

This approximation, however, is not valid if $\tau_2 = 0$ (or $v_2 = 0$), in which case the searcher performs a simple diffusion. Thus we assume that τ_2 is definitely smaller than τ_1, but of the same order. A more precise discussion of this point is given in appendix A. Then we have, with the notations of formulae (5)–(7)

$$\begin{aligned} c_{2n+1} - b_1 &\sim c_{2n+1} - x_{2n} + \sum_{1\leqslant i\leqslant n} (x_{2i} - x_{2i-1}) + \sum_{1\leqslant i\leqslant n-1} (x_{2i+1} - x_{2i}) + x_0 - b_1 \\ &= \chi_{2n+1} + \sum_{1\leqslant i\leqslant n} y_{2i} + \sum_{1\leqslant i\leqslant n-1} y_{2i-1} - \beta_1. \end{aligned}$$

Using these approximations, and assuming that the waiting times in both the diffusive and the ballistic phases are exponential, it is shown in appendix A that the mean search time is given by

$$\langle T\rangle = (\tau_1 + \tau_2)\left\{\frac{1 - \nu(D\tau_1)^{-1/2}}{\nu v\tau_2 - \nu^2 D\tau_1 - \nu^3 v\tau_2 D\tau_1} + k(0,\nu)\right\}, \tag{31}$$

$k(0, \nu)$ being given by (25). Because of inequality (11) the approximation of small ballistic displacements, based on approximation (30), should lead to a *lower bound* of the search time $\langle T\rangle$, as was the case for the approximation of large ballistic displacements. For obtaining the explicit expression (31) of $\langle T\rangle$, however, we had to replace (30) by the larger estimation (30′), so that (31) is not rigorously a lower bound of the search time, but practically, this should be the case in most situations.

The approximation (31) breaks down if $v\tau_2 - \nu D\tau_1 - \nu^2 v\tau_2 D\tau_1 < 0$ (see appendix A). We now focus on the case of very small densities, such that $\nu\, v\tau_2 \ll 1$: then the approximation holds if $\nu D\tau_1 < v\tau_2$. Then the term $k(0, \nu)$ in the right-hand side of equation (31) can be neglected with respect to the other one and we have

$$\langle T\rangle \sim (\tau_1 + \tau_2)\frac{1}{\nu v\tau_2}. \tag{32}$$

Thus, in the present approximation of small ballistic displacements, the mean search time $\langle T\rangle$ is a decreasing function of τ_2, when τ_1 is maintained constant, whereas it was an increasing function of τ_2 in the approximation of large ballistic displacements, both approximations (27) and (32) being lower bounds of $\langle T\rangle$. As a consequence, when τ_1 is kept constant and τ_2 increases from 0 to infinity, $\langle T\rangle$ first decreases for $v\tau_2 \ll (D\tau_1)^{1/2}$ (except, perhaps, in the neighbourhood of 0: see the discussion of approximation (31)), then increases for $v\tau_2 \gg (D\tau_1)^{1/2}$, so that we can deduce the important conclusion that $\langle T\rangle$ *necessarily has* (*at least*) *one minimum* as a function of τ_2. The minimum should presumably occur for $v\tau_2 \sim 2(D\tau_1)^{1/2}$, for which value both lower bounds coincide. This common value can be written as

$$\langle T\rangle \sim \frac{L}{v}\left[1 + \frac{1}{2}(\tau_1/\tau)^{1/2}\right], \tag{33}$$

where we used the characteristic time $\tau = Dv^{-2}$ during which the ballistic displacement and the diffusive mean square displacement are of the same order. However, we cannot expect formula (33) to give an accurate estimation of the minimum value of $\langle T\rangle$, since it is obtained at the limit of validity for both approximations. Thus we now consider a third approximation, in the intermediate case of medium ballistic displacements.

3.5. Intermediary approximation

If the average displacement $v\tau_2$ during a ballistic phase is comparable to the span $(2D\tau_1)^{1/2}$ of a diffusive phase, the intervals scanned during successive diffusive phases may or may not be disjoint with finite probabilities, and both previous approximations break down. However, it is possible to make an intermediary approximation, by considering the probability $\pi(t|y, \chi)$ that during time t a diffusive phase has no common point with the previous diffusive phase, which scanned the interval (β, χ) around the initial position 0 and final position y of the searcher. The mean value p of $\pi(t|y, \chi)$, averaged on the trajectory of the searcher during the two successive phases and on the durations of these phases, can be estimated (see appendix B) by

$$p = \left[\frac{v\tau_2}{v\tau_2 + (D\tau_1)^{1/2}}\right]^2.$$

It is shown in appendix B that this probability allows improving the previous approximations used to compute the averages on the frozen disorder. The resulting mean search time is, in the limit of small target densities when $\varepsilon \equiv \nu(D\tau_1)^{1/2} \ll 1$

$$\langle T\rangle \sim \frac{\tau_1 + \tau_2}{\nu\, v\tau_2}\frac{(1+\theta)^2(1+\varepsilon\theta)}{1 + 4\theta + 2\varepsilon\theta^2} \tag{34}$$

with $\theta = v\tau_2/(D\tau_1)^{1/2}$, which correctly yields the limit behaviours (27) and (32) when $\tau_2 \to \infty$ and $\tau_2 \to 0$, respectively. Formula (34) shows that in the limit of low target densities, the mean search time $\langle T \rangle$ again scales as $\nu^{-1} = L$, the average distance between two neighbouring targets, which allows us to conclude that intermittency is favourable for minimizing the search time.

Furthermore, it can be shown (appendix B) that for a given τ_1, $\langle T \rangle$, as given by (34), decreases with τ_2 for small values of τ_2, but increases for large values of τ_2, and has a minimum value. This minimum value satisfies the following scaling laws [1], written with the reduced times $\tau_1 = \tau_1/\tau$, $\tau_2 = \tau_2/\tau$:

$$\bullet \text{ If } \tau_1 \ll 1, \quad \text{we have} \quad \tau_2 \gg \tau_1 \quad \text{and} \quad \tau_2 \sim \tfrac{1}{2}(\tau_1)^{1/2}. \tag{35}$$

$$\bullet \text{ If } \tau_1 \gg 1, \quad \text{we have} \quad \tau_2 \ll \tau_1 \quad \text{and} \quad \tau_2 \sim (7/4)^{1/2}(\tau_1)^{3/4}. \tag{35$'$}$$

A global study of $\langle T \rangle$ shows that the minimum possible value of $\langle T \rangle$ is $\langle T \rangle_0 = \frac{3}{4}L/v$, which can only be obtained when τ_1 and τ_2 both tend to 0. It can be concluded, as in the case of a periodic distribution of targets [22], that in order to minimize the search time, it is necessary to choose τ_1 and τ_2 as small as possible, which is far from being obvious. However, in practical cases, there is generally a lower bound τ_{1m} for the time τ_1 of the diffusive phase, in order to allow for the detection of a target. Then, the optimal search strategy is to choose for τ_1 this minimum value τ_{1m}, and then to choose τ_2 in agreement with the laws (35) or (35′).

4. Discussion and comparison with numerical simulations

4.1. Discussion: efficiency of intermittence and gain

For this discussion it is convenient to introduce the characteristic times of the ballistic and diffusive regimes, defined respectively by $\tau_{\text{ball}} = L/v$ and $\tau_{\text{diff}} = L^2/D$, the global characteristic time of the process τ being

$$\tau \equiv \frac{D}{v^2} = \frac{(\tau_{\text{ball}})^2}{\tau_{\text{diff}}}. \tag{36}$$

It is important to characterize the efficiency of intermittence, as compared with a purely diffusive motion. It can be estimated by the gain G defined by

$$G = \frac{\langle T \rangle_{\text{diff}}}{\langle T \rangle_{\text{min}}}, \tag{37}$$

where $\langle T \rangle_{\text{min}}$ is the minimum value of the search time, obtained by the optimal choice of parameters τ_1 and τ_2, and $\langle T \rangle_{\text{diff}}$ is the search time (26) when the system is always in its 'reactive' regime.

When the approximation of large ballistic displacements holds, we should have $v\tau_2 \gg (D\tau_1)^{1/2}$, or $\tau_2 \gg (\tau\tau_1)^{1/2}$. Then it is seen from (28) and (26) that

$$\frac{\langle T \rangle_{\text{diff}}}{\langle T \rangle} \sim \frac{(\tau_1 \tau_{\text{diff}})^{1/2}}{\tau_1 + \tau_2}. \tag{38}$$

This ratio is very large if $(\tau_1\tau_{\text{diff}})^{1/2} \gg \tau_1 + \tau_2 \gg \tau_1 + (\tau\tau_1)^{1/2}$, which can be realized if $\tau_{\text{diff}} \gg \tau$, or by (36), $\tau_{\text{diff}} \gg \tau_{\text{ball}}$: this is again condition (29).

On the other hand, in the approximation of small ballistic displacements, it is easily seen from (32) that $\langle T \rangle \geqslant \langle T \rangle_{\text{diff}}$, so that intermittence is *not an efficient strategy* in this case, as could be anticipated: it is better, if possible, to stay forever in the diffusive phase.

Eventually, in the case of the intermediary approximation, if τ_1 is fixed it is shown in appendix A that, as mentioned in section 3.3, the search time has a minimum $\langle T \rangle_{\text{min}}$ with respect to τ_2, depending on the situation:

- if $\tau_1 \gg \tau = D/v^2$, we have

$$\langle T\rangle_{\min} \sim \frac{1}{2}\frac{L}{v}\left(\frac{\tau_1}{\tau}\right)^{1/2} \tag{39}$$

  and the gain is

$$G \sim \frac{L}{(D\tau_1)^{1/2}} = \left(\frac{\tau_{\text{diff}}}{\tau_1}\right)^{1/2}. \tag{40}$$

  It can be very large for low densities, and more precisely, if $\tau \ll \tau_1 \ll \tau_{\text{diff}}$, which implies by (37) $\tau_{\text{diff}}/\tau = (\tau_{\text{ball}}/\tau)^2 \gg 1$, and again leads to $\tau_{\text{diff}} \gg \tau_{\text{ball}}$.
- if $\tau_1 \ll \tau = D/v^2$, we have

$$\langle T\rangle_{\min} \sim \frac{3}{4}\frac{L}{v} = \frac{3}{4}\tau_{\text{ball}}. \tag{41}$$

This is even less than the search time that would be expected in a purely ballistic search (if ballistic motion would permit to find the target)! It should be recalled, however, that simple diffusion, based on Brownian motion, is only a 'slow' motion for large times. When time tends to 0, its 'velocity' becomes infinite and this model may be unphysical for certain applications. Using, nevertheless, formula (41) yields the gain

$$G \sim \frac{2}{3}\frac{Lv}{D} = \frac{2}{3}\frac{\tau_{\text{diff}}}{\tau_{\text{ball}}}, \tag{42}$$

which is very large for low target densities. In principle (see appendix B) this last value is the upper bound that can be expected for the gain, but it can be difficult to obtain it in practise, or even, this limit may be physically irrelevant, as commented above.

It is seen that in all cases, condition (29), or equivalently

$$\tau_{\text{diff}} \gg \tau_{\text{ball}}, \tag{43}$$

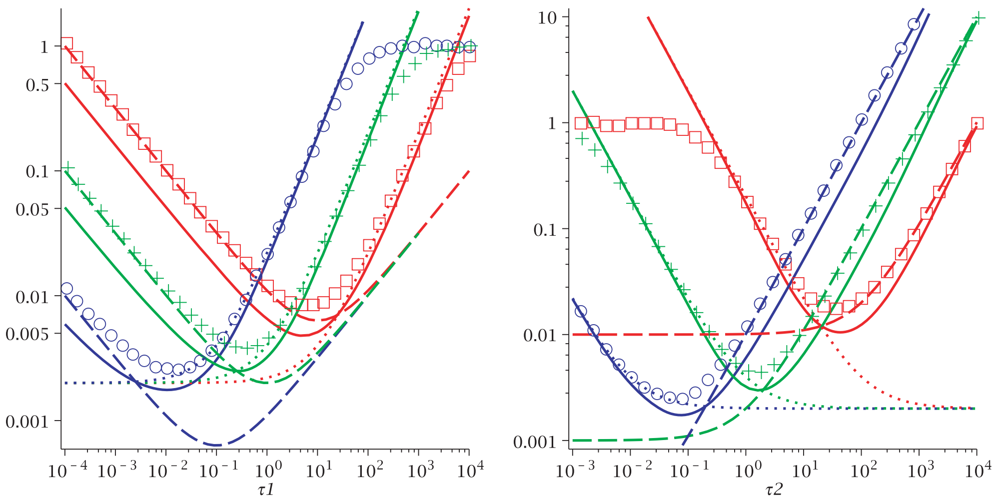
which is obviously necessary in order that intermittence may be favourable for minimizing the search time, indeed allows one to obtain a large gain from intermittent behaviour by an adequate tuning of the parameters.

5. Comparison with numerical simulations

The previous intermediary approximation is based on a coarse evaluation of the correlation between the intervals explored during two successive diffusive phases. Thus the resulting estimations of $\langle T\rangle$, although reasonable, should be compared with numerical simulations.

Figure 1 represents the mean search time $\langle T\rangle$ in functions of τ_1 and τ_2 for typical values of the other parameters. They allow comparing the numerical results with the approximations (28) and (32), and with the intermediary approximation (34). It is seen that the approximations of large and small ballistic displacements are valid in the expected conditions, and that they indeed yield lower bounds for $\langle T\rangle$. On the other hand, the intermediary approximation (34) correctly reproduces the existence and the position of the minimum of $\langle T\rangle$. Moreover, it provides an accurate estimation of the search time if τ_1 is much larger than the characteristic time $\tau = D/v^2$, and if τ_2 is of the order or greater than τ. The approximations break down if $\tau_2/\tau_1 < Lv/D$, as was expected.

Figure 2 qualitatively supports the scaling laws relating τ_1 and the corresponding optimal waiting time τ_2. The exponent 3/4 of the theoretical scaling law (35′) for $\tau \gg \tau_1$ is very well confirmed by the simulations. This is not the case for law (35) for $\tau_1, \tau_2 \ll \tau$, which again



$ln(\langle T\rangle/T_{diff})$ as a function of $ln(\tau_1)$. $\tau_2 = 10$ (red, squares), $\tau_2 = 1$ (green, crosses), $\tau_2 = 0.1$ (blue, circles).

$ln(\langle T\rangle/T_{diff})$ as a function of $ln(\tau_2)$. $\tau_1 = 100$ (red, squares), $\tau_1 = 1$ (green, crosses), $\tau_1 = 0.01$ (blue, circles).

Figure 1. Validity of the approximations. Small ballistic displacements (32) (dotted line). Large ballistic displacements (28) (small dots). Intermediary approximation (34) (line). Numerical simulations (points): $D = 1$, $v = 1$, $\nu = 10^{-3}$.

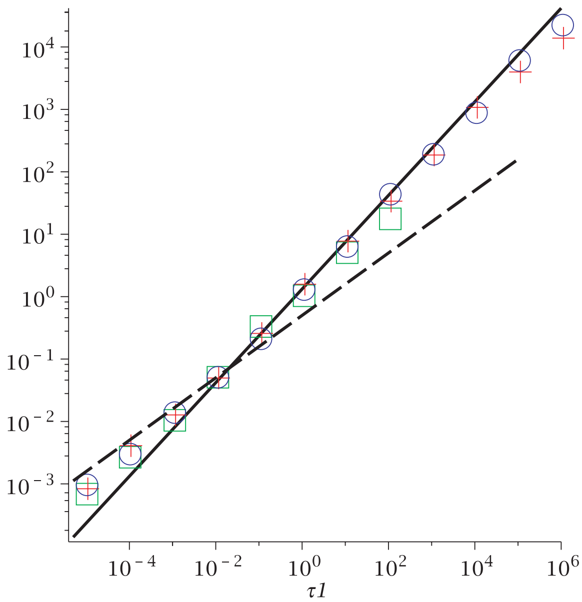


Figure 2. Ln (τ_2^{opt}) as a function of ln(τ_1). Small τ_1 analytical prediction (35) (dotted black line). Large τ_1 analytical prediction (35′) (dotted black line). Numerical values (points), for $\nu = 10^{-1}$ (green squares), $\nu = 10^{-3}$ (red crosses), $\nu = 10^{-5}$ (blue circles). $D = 1$, $v = 1$.

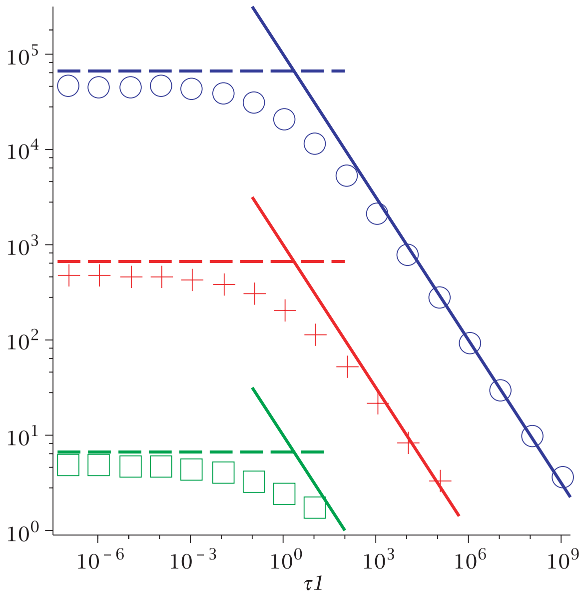


Figure 3. Ln(G) as a function of ln(τ_1) (τ_2 taken optimal). Small τ_1 analytical prediction (42) (dotted line). Large τ_1 analytical prediction (40) (line). Numerical simulations (points): $\nu = 10^{-1}$ (green, squares), $\nu = 10^{-3}$ (red, crosses), $\nu = 10^{-5}$ (blue, circles). $D = 1$, $v = 1$.

indicates that the approximations should be handled with care for short waiting times τ_1, τ_2, although their results are qualitatively correct.

Figure 3 shows the gain in the function of τ_1 in different possible conditions. It supports the conclusions of the theoretical study, and indeed confirms that the gain due to intermittence can be very important if condition (43) is satisfied.

6. Conclusion

We have systematically studied intermittency in a disordered one-dimensional medium, when a searcher can either scan its domain according to a diffusive motion, or undergo a ballistic, non-reactive displacement, till it discovers one of Poisson distributed targets. Partial theoretical results, previously presented, have been confirmed, completed and, for the first time, derived. We have also extended significantly the conditions of this study by considering the possibility of non-exponential waiting times in each of the dynamical regimes. We have shown that alternating the two dynamical regimes can allow one to decrease the mean search time with respect to a search based on a purely diffusive motion. The mean search time has been computed explicitly with the aid of a simple model, and the optimal conditions have been obtained as well. Thus we showed that the waiting times in each regime play a basic role as control parameters in this optimization problem, and we found that their optimal values are related by scaling laws, as is the case for regularly spaced targets, but with different scaling exponents.

Numerical simulations have been realized: they confirm the theoretical results qualitatively, with good quantitative agreement in most cases. The most remarkable point is that both the theoretical and the numerical analyses show that intermittent behaviour can allow for minimizing the search time in a disordered medium and obtain a large gain in comparison with the purely diffusive exploration which intuitively appears to be more reasonable. This conclusion can have important consequences for analysing many natural phenomena, designing technological devices or organizing industrial or social processes. Thus it would be justified to extend the present model to other kinds of disorder or to higher dimensions, in order to have an adequate representation of a larger class of phenomena. In particular, other laws for the waiting time in the ballistic regime should be investigated. In fact, examples of intermittency considered in several fields show that similar waiting times can be strongly non-exponential. For instance, they are often much better, if not exactly, represented by a heavy-tailed law (as reported and discussed, for instance, in [10, 39] or [40]) or by a γ law [39]; on the other hand, if the searcher uses a voluntary strategy or is tributary of rational, internal factors (such as, for instance, fatigue or energy expenses), the waiting time in the displacement regime should be more or less deterministic. Such developments, based on the techniques described in the present paper, are under work.

Acknowledgment

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Appendix A.

A.1. Small displacements approximation

Thanks to formula (30), the Laplace transform of the survival probability after $2n$ changes of regime, equation (6), can be written for $n \geqslant 1$:

$$\begin{aligned}\tilde{S}_{2n}(s|x_0) = &\int \mathrm{d}y\,\mathrm{d}\beta_{2n+1}\,\mathrm{d}\chi_{2n+1}\,Q_1(s, y, \beta_{2n+1}, \chi_{2n+1})\exp(-\nu\chi_{2n+1}).\\ &\cdot\int\prod_{1\leqslant k\leqslant n}\mathrm{d}y_{2k}q_2(s, y_{2k})\exp(-\nu y_{2k})\\ &\cdot\int\prod_{1\leqslant k\leqslant n-1}\mathrm{d}y_{2k-1}\,\mathrm{d}\beta_{2k-1}\mathrm{d}\chi_{2k-1}q_1(s, y_{2k-1}, \beta_{2k-1}, \chi_{2k-1})\exp(-\nu\, y_{2k-1})\\ &\cdot\int\mathrm{d}y_1\,\mathrm{d}\beta_1\mathrm{d}\chi_1 q_1(s, y_1, \beta_1\chi_1)\exp(-\nu(y_1-\beta_1))\end{aligned}\tag{A.1}$$

with $-\infty < \beta_{2k+1} < 0$, $0 < \chi_{2k+1} < +\infty$, $\beta_{2k+1} < y_{2k+1} < \chi_{2k+1}$, and $-\infty < y_{2k} < \infty$.

Let us write, in analogy with (14)

$$\begin{aligned}q_2(s,\nu) &\equiv \int \mathrm{d}y\, q_2(s,y)\exp(-\nu y), \qquad Q_2(s) \equiv \textstyle\int\, \mathrm{d}y\, Q_2(s,y)\\ h(s,\nu,0,0) &= \int_{-\infty<\beta<0}\mathrm{d}\beta\int_{0<x<\infty}\mathrm{d}\chi\int_{\beta<y<\chi}\mathrm{d}y\, q_1(s,y,\beta,\chi)\exp(-\nu y)\\ h(s,\nu,\nu,0) &= \int_{-\infty<\beta<0}\mathrm{d}\beta\int_{0<x<\infty}\mathrm{d}\chi\int_{\beta<y<\chi}\mathrm{d}y\, q_1(s,y,\beta,\chi)\exp(-\nu(y-\beta))\\ H(s,0,0,\nu) &= \tau_1\, h(s,0,0,\nu) = \int_{-\infty<\beta<0}\mathrm{d}\beta\int_{0<x<\infty}\mathrm{d}\chi\int_{\beta<y<\chi}\mathrm{d}y\, Q_1(s,y,\beta,\chi)\exp(-\nu\,\chi)\\ H(s,0,\nu,\nu) &= \tau_1 h(s,0,\nu,\nu)\int_{-\infty<\beta<0}\mathrm{d}\beta\int_{0<x<\infty}\mathrm{d}\chi\int_{\beta<y<\chi}\mathrm{d}y\, Q_1(s,y,\beta,\chi)\\ &\quad\times\exp(-\nu(\chi-\beta)).\end{aligned}\tag{A.2}$$

With these notations, equation (A.1) gives

$$\tilde{S}_{2n}(s|x_0) = H(s,0,0,\nu)[q_2(s,\nu)]^n\,[h(s,\nu,0,0)]^{n-1}h(s,\nu,\nu,0) \qquad (n>0) \tag{A.3}$$

$$\tilde{S}_0(s|x_0) = H(s,0,\nu,\nu) \equiv K(s,\nu). \tag{A.4}$$

Similarly it is found that

$$\tilde{S}_{2n+1}(s|x_0) = Q_2(s)\,h(s,0,0,\nu)\,[q_2(s,\nu)]^n\,[h(s,\nu,0,0)]^{n-1}\,h(s,\nu,\nu,0) \quad (n>0) \tag{A.5}$$

$$\tilde{S}_1(s|x_0) = Q_2(s)\,h(s,0,\nu,\nu). \tag{A.6}$$

Summing on n we obtain $\tilde{S}(s) \equiv \tilde{S}\,(s|x_0)$

$$\tilde{S}(s) = \frac{[H(s,0,0,\nu) + Q_2(s)h(s,0,0,\nu)]q_2(s,\nu)h(s,\nu,\nu,0)}{1 - q_2(s,\nu)h(s,\nu,0,0)} + K(s,\nu) + Q_2(s)h(s,0,\nu,\nu) \tag{A.7}$$

provided that $q_2(s,\nu)h(s,\nu,0,0) < 1$. Otherwise, the sum of $\tilde{S}_k(s|x_0)$ does not converge and the whole approximation breaks down.

$H(s,0,0,\nu)$ is the Laplace transform of the probability that at time t the phase 1 has not finished and that the searcher has found no target on its right-hand side, which is not greater than the survival probability $\phi_1(t)$ of phase 1 at t. Thus $sH(s,0,0,\nu) < s\,\tilde{\phi}_1(s) \to 0$ if $\phi_1(\infty) = 0$. Furthermore, it will be seen that $h(s,\nu,0,0)$ remains finite when $s \to 0$. On the other hand, if $s \to 0$, $sQ_2(s) = 1 - q_2(s) \to 0$, and it has been shown previously that $sK(s,\nu) \to 0$ if $s \to 0$ Thus it can be concluded that, as expected, the survival probability vanishes when $t \to \infty$

$$S(\infty) = \lim_{s\to 0} s\tilde{S}(s) = 0.$$

It is seen that $\tilde{S}(s)$ and the average search time $\langle T\rangle = \tilde{S}(0)$ now depend on the precise laws of the waiting times T_i in phases $i = 1$ and 2. In order to obtain an explicit expression for $\langle T\rangle$, we now assume that both waiting times T_i are exponential: $P\,(T_i > t) = \exp(-\lambda_i\,t) = \exp(-t/\tau_i)$, $i = 1, 2$.

Then it is easily found that

$$q_2(0,\nu) = \int_{0<y<\infty} \mathrm{d}y\,\mathrm{e}^{-\nu y} \int_{0<t<\infty} \mathrm{d}t\,\delta(y - vt)\lambda_2\exp(-\lambda_2 t) = (1 + \nu v\tau_2)^{-1}. \tag{A.8}$$

Furthermore we have

$$\begin{aligned} h(s,\nu,0,0) &= \int_{-\infty<\beta<0} \mathrm{d}\beta \int_{0<x<\infty} \mathrm{d}\chi \int_{\beta<y<\chi} \mathrm{d}y\,q_1(s,y,\beta,\chi)\exp(-\nu y) \\ &= \lambda_1 \int_{-\infty<y<\infty} \mathrm{d}y\,\tilde{p}_1(s+\lambda_1,y)\exp(-\nu y) \end{aligned}$$

with $p_1(t,y)$ being the free diffusion propagator, and

$$\begin{aligned} &\int_{-\infty<y<\infty} \mathrm{d}y\,p_1(t,y)\exp(-\nu y) \\ &\quad = \int_{-\infty<y<\infty} \mathrm{d}y\,(4\pi Dt)^{-1/2}\exp(-y^2/(4Dt) - \nu y) = \exp(D\nu^2 t) \end{aligned}$$

so that the Laplace transform $h(s,\nu,0,0)$ is finite if $s + \lambda_1 - D\nu^2 > 0$. Thus, if $\lambda_1 > D\nu^2$

$$h(0,\nu,0,0) = \lambda_1(\lambda_1 - D\nu^2)^{-1} = (1 - \nu^2 D\,\tau_1)^{-1}. \tag{A.9}$$

If, in contrast, $\nu(D\tau_1)^{1/2} > 1$, $h(s, \nu, 0, 0)$ becomes infinite and the present formalism does not apply. From now on, we suppose that $\nu(D\tau_1)^{1/2} < 1$. Let us now consider

$$H(s,0,0,\nu) = \int_{-\infty<\beta<0} \mathrm{d}\beta \int_{0<x<\infty} \mathrm{d}\chi \int_{\beta<y<\chi} \mathrm{d}y\, Q_1(s, y\, \beta, \chi) \exp(-\nu\,\chi)$$
$$\times \int_{0<x<\infty} \mathrm{d}\chi \int_{\beta<y<\chi} \mathrm{d}y\, \tilde{p}_1(s+\lambda_1, y, \chi) \exp(-\nu\,\chi), \tag{A.10}$$

where $p_1(t, y, \beta)$ is the probability density to be at position y at time t, with $\sup_{0<t'<t} y(t') = \chi$, starting from $y = 0$ at time 0. Relation (21) is now replaced by

$$p_1(t, y, \chi) = -\partial/\partial\chi p_1(t, y|\chi), \tag{A.11}$$

where $p_1(t, y|\chi)$ is the probability to be at y at time y, knowing that χ is an absorbing point.

Integrating (A.10) by parts yields

$$H(s,0,0,\nu) = \nu \int_{0<\chi<\infty} \mathrm{d}\chi \exp(-\nu\chi)\ \tilde{p}_1(s+\lambda_1|\chi),$$

with $p_1(t|\chi)$ being the survival probability at time t, starting from $y = 0$ at $t = 0$, knowing that $\chi > 0$ is an absorbing point. It is known that its Laplace transform is

$$\tilde{p}_1(s|\chi) = (1 - \exp(-(s/D)^{1/2}\chi))/s, \tag{A.12}$$

which gives

$$H(0,0,0,\nu) = \tau_1 \frac{1}{1+\nu(D\,\tau_1)^{1/2}}. \tag{A.13}$$

We now compute

$$h(s,\nu,\nu,0) = \int_{-\infty<\beta<0} \mathrm{d}\beta \int_{0<x<\infty} \mathrm{d}\chi \int_{\beta<y<\chi} \mathrm{d}y\, q_1(s, y, \beta, \chi) \exp(-\nu(y-\beta))$$
$$= \lambda_1 \int_{-\infty<\beta<0} \mathrm{d}\beta \int_{\beta<y<\chi} \mathrm{d}y\, \tilde{p}_1(s+\lambda_1, y, \beta) \exp(-\nu(y-\beta)), \tag{A.14}$$

where $p_1(t, y, \beta)$ is the probability density to be at position y at time t, with $\inf_{0<t'<t} y(t') = \beta$, starting from $y = 0$ at time 0. Relation (A.11) is replaced by

$$p_1(t, y, \beta) = (\partial/\partial\beta)\, p_1(t, y|\beta), \tag{A.15}$$

where $p_1(t, y|\beta)$ is the probability to be at y at time t, starting from 0 at time 0, knowing that β is an absorbing point. Integrating (A.14) by parts yields

$$h(s,\nu,\nu,0) = \lambda_1\nu \int_{-\infty<\beta<0} \mathrm{d}\beta \exp(\nu\beta) \int_{\beta<y<\infty} \mathrm{d}y\, \tilde{p}_1(s+\lambda_1, y|\,\beta) \exp(-\nu y). \tag{A.16}$$

The Laplace transform of $p_1(t, y|\beta)$ is easily found to be [38, 39]

$$\tilde{p}_1(s, y|\,\beta) = \tfrac{1}{2}(sD)^{-1/2}\,[\exp((s/D)^{1/2}y) - \exp((s/D)^{1/2}\,(2\beta - y))] \quad \text{if } \beta < y < 0$$
$$= \tfrac{1}{2}(sD)^{-1/2}[\exp(-(s/D)^{1/2}y) - \exp((s/D)^{1/2}(2\beta - y))] \quad \text{if } 0 < y. \tag{A.17}$$

Writing again $\alpha = (\lambda_1/D)^{1/2} = (D\,\tau_1)^{-1/2}$, the integral over y in the right-hand side of (A.16) yields for $s = 0$

$$\frac{1}{\alpha-\nu}[1 - \mathrm{e}^{(\alpha-\nu)\beta}] + \frac{1}{\alpha+\nu} - \mathrm{e}^{2\alpha\beta}\frac{1}{\alpha+\nu}\,\mathrm{e}^{-(\alpha+\nu)\beta} = \frac{2\alpha}{\alpha^2-\nu^2}[1 - \mathrm{e}^{(\alpha-\nu)\beta}].$$

The integration over β in the right-hand side of (A.16) eventually gives

$$h(0,\nu,\nu,0) = \frac{\lambda_1}{2(D\lambda_1)^{1/2}}\frac{2\alpha\nu}{\alpha^2-\nu^2}\left[\frac{1}{\nu} - \frac{1}{\alpha}\right] = \frac{1}{1+\nu(D\tau_1)^{1/2}}. \tag{A.18}$$

Inserting these partial results into equation (A.7), we obtain the average search time $\langle T\rangle = \tilde{S}(0)$

$$\langle T\rangle = (\tau_1+\tau_2)\left\{\frac{1-\nu(D\tau_1)^{1/2}}{\nu v\tau_2 - \nu^2 D\tau_1 - \nu^3 v\tau_2 D\tau_1} + k(0,\nu)\right\}, \tag{A.19}$$

with $k(0, \nu)$ being given by (25). It is clear that the approximation is only valid if $v\tau_2 - \nu D\tau_1 - \nu^2 v\tau_2 D\tau_1 > 0$. This is realized if

$$\frac{D\tau_1}{L^2} < \frac{\frac{v\tau_2}{L}}{1+\frac{v\tau_2}{L}} \tag{A.20}$$

which implies in particular $D\tau_1/L^2 > v\tau_2/L$. In the case of low target density, $\nu(D\tau_1)^{1/2} \ll 1$, condition (A.20) reduces to

$$\frac{D\tau_1}{L^2} < \frac{v\tau_2}{L} \ll 1, \qquad \text{or} \qquad \frac{\tau_1}{\tau_{\text{diff}}} < \frac{\tau_2}{\tau_{\text{ball}}} \ll 1, \tag{A.21}$$

where $\tau_{\text{ball}} = L/v$ and $\tau_{\text{diff}} = L^2/D$ are the characteristic times of the ballistic and diffusive regimes. From now on, we focus on this case. Then the term $k(0, \nu)$ in the right-hand side of equation (A.19) can be neglected with respect to the other one and we have

$$\langle T\rangle \sim (\tau_1+\tau_2)\frac{1}{\nu v\tau_2}. \tag{A.22}$$

Appendix B. Intermediary approximation: medium ballistic displacements

B.1. Calculation of the mean search time

If the average displacement $v\tau_2$ during a ballistic phase is comparable to the span $(2D\tau_1)^{1/2}$ of a diffusive phase, the intervals scanned during successive diffusive phases may or may not be disjoint with finite probabilities, and both previous approximations break down. For the sake of simplicity, let us assume that the ballistic phases have a constant, deterministic duration t_2. We define the probability $\pi(t|y, \chi)$ that during time t a diffusive phase has no common point with the previous diffusive phase, which scanned the interval (β, χ) around the initial position 0 and final position y of the searcher. Writing $\beta' = \chi - y - v\,t_2$, we have

$$\pi(t|y,\chi) = 0 \qquad \text{if} \quad \beta' = \chi - y - vt_2 = 0.$$

If $\beta' = \chi - y - vt_2 < 0$, $\pi(t \mid y, \chi)$ is the survival probability $p_1(t|\beta')$ at time t of a diffusive motion starting from 0, with an absorbing point at β': $\pi(t|y, \chi) = p_1(t|\beta')$. Its average value at the end of regime 1 is, according to the notations of appendix A,

$$\lambda_1\tilde{\pi}(\lambda_1|y,\chi) = \lambda_1\,\tilde{p}_1(\lambda_1|y+vt_2-\chi) = 1-\exp[-(D\tau_1)^{-1/2}(y+vt_2-\chi)] \tag{B.1}$$

Here $\xi = y - \chi < 0$ is the minimum value, between times 0 and t, of a diffusion starting from 0 at time 0. The probability density of ξ is $p_1(\xi, t) = -\partial/\partial\xi\; p_1(t|\xi)$ (because $\xi < 0$) and its average value with respect to the duration of the diffusive phase is

$$\lambda_1\tilde{p}_1(\lambda_1,\xi) = -\partial/\partial\xi[1-\exp((D\tau_1)^{-1/2}\xi)] = (D\tau_1)^{-1/2}\exp((D\tau_1)^{-1/2}\xi).$$

The average value of $\pi(t|y, \chi)$ can be estimated by

$$\int_{-v\tau 2<\xi<0} \mathrm{d}\xi[1-\exp[-(D\tau_1)^{-1/2}(\xi+vt_2)]\,(D\tau_1)^{-1/2}\exp((D\tau_1)^{-1/2}\xi)$$
$$= 1-\exp[-vt_2(D\tau_1)^{-1/2}] - v\tau_2(D\tau_1)^{-1/2}\exp[-vt_2(D\tau_1)^{-1/2}].$$

If now the duration t_2 of the ballistic regime is exponentially distributed with mean value τ_2, a final average over this duration gives the mean probability that two successive diffusive regimes are independent

$$p = \left[\frac{v\tau_2}{v\tau_2 + (D\tau_1)^{1/2}}\right]^2. \tag{B.2}$$

Now, let us consider a sequence of $2N + 1$ changes of regime between time 0 and t, starting from the diffusive regime 1, so that the searcher is in the ballistic regime 2 at time t. Suppose that the following situation holds:

- There are first n_1 ($n_1 \geqslant 0$) successive alternations of diffusive regimes and ballistic regimes such that the intervals scanned during these diffusive regimes are independent (they have no common points).
- Then follow n_2 ($n_2 \geqslant 2$) successive alternations diffusive regime–ballistic regime such that the intervals scanned during two diffusive regimes are interconnected (they have a non-empty intersection).
- Then follow ($n_3 \geqslant 1$) independent alternations of diffusive regime–ballistic regime, etc.
- Finally, we have n_{2m+1} ($n_{2m+1} \geqslant 0$) independent alternations of diffusive regime–ballistic regime,

with $n_1 + n_2 + \cdots + n_{2m+1} = 2N + 1$. By making n_1 or n_{2m+1} equal to 0, we can consider situations where the searcher begins or finishes with interconnected alternations of diffusive and ballistic regimes. The probability of such a sequence can be coarsely estimated by means of the average probability p that two successive diffusive phases are independent, equation (B.2). It is given by

$$\Pi(n_1, n_2, n_{2m+1}) \equiv p^{n_1} \cdot q^{n_2-1} p \cdot p^{n3} \cdot q^{n_4-1} p \cdots p^{n_{2m+1}-1}.$$

The Laplace transform of the average survival probability of the searcher during these $2N + 1$ changes of regimes is, from formula (7), (9), (14) and (A.5),

$$\tilde{S}_{2N+1}(s) = \tau_2[f(s,\nu)]^{n_1} c(s,\nu)[g(s,\nu)]^{n_2-2}[f(s,\nu)]^{n_3} c(s,\nu)[g(s,\nu)]^{n_4-2} \cdots [f(s,\nu)]^{n_{2m+1}} \tag{B.3}$$

with

$$f(s,\nu) = q_2(s)\int_{-\infty<\beta<0} \mathrm{d}\beta \int_{0<x<\infty} \mathrm{d}\chi \int_{-\beta<y<\chi} \mathrm{d}y\, q_1(s+\lambda_1, y, \beta, \chi) \times \exp(-\nu(\chi-\beta)) \equiv \tilde{q}_2(s)k(s,\nu), \tag{B.4}$$

where $q_2(s) = \lambda_2\,(s+\lambda_2)^{-1}$ and

$$c(s,\nu) = h(s,0,0,\nu)q_2(s,\nu)h(s,\nu,\nu,0); \qquad g(s,\nu) = q_2(s,\nu)\,h(s,\nu,0,0). \tag{B.5}$$

For a given m the average of $\Sigma_{N\geqslant 0}\,\tilde{S}_{2N+1}(0)$ is obtained by summing $\tilde{S}_{2N+1}(0)\;\Pi(n_1, n_2, n_{2m+1})$ over $n_1 \geqslant 0$, $n_2 \geqslant 2$ $n_3 \geqslant 0,\ldots, n_{2m+1} \geqslant 0$. Different contributions can be distinguished:

(a) if $m \geqslant 0$, $n_1 \geqslant 1, \ldots, n_{2m} \geqslant 2$, $n_{2m+1} \geqslant 0$, we obtain

$$X_a = \tau_2 \sum_{n_1\geqslant 1}\sum_{n_2\geqslant 2}\sum_{n_3\geqslant 0}\cdots\sum_{n_{2m+1}\geqslant 0} (pf)^{n_1}(qg)^{n_2-2}cpq(pf)^{n_3}cq\cdots(pf)^{n_{2m+1}}p^{-1}$$
$$= \tau_2\left(\frac{pf}{1-pf}\right)\left(\frac{1}{1-pf}\right)^m\left(\frac{1}{1-qg}\right)^m(cpq)^m p^{-1}$$

(b) if $m > 0, n_1 = 0, n_2 \geqslant 2, \ldots, n_{2m} \geqslant 2, n_{2m+1} \geqslant 0$,

$$X_b = \tau_2 \sum_{n_2\geqslant 2}\sum_{n_3\geqslant 0}\cdots\sum_{n_{2m}\geqslant 2}\sum_{n_{2m+1}\geqslant 1}(qg)^{n_2-2}cpq(pf)^{n_3}\cdots(qg)^{n_{2m}-2}cpq(pf)^{n_{2m+1}}p^{-1}$$

$$= \tau_2\left(\frac{1}{1-pf}\right)^m\left(\frac{1}{1-qg}\right)^m(cpq)^m p^{-1} = X_b.$$

We now sum up these contributions over all m, supposing that all geometrical series are convergent, which is the case for low target densities ν. We eventually obtain

$$\sum_{N\geqslant 0}\tilde{S}_{2N+1}(0) = \tau_2\frac{\frac{k}{1-pk}+\frac{qc}{(1-pk)(1-qg)}}{1-\frac{pqc}{(1-pk)(1-qg)}} \tag{B.6}$$

with $p = 1-q$ given by (B.2), and according to (25) and (A.9)–(A.18) if $\nu(D\tau_1)^{1/2} \ll 1$

$$\begin{aligned} &k \equiv k(0,\nu) \sim 1 - 2\,\nu(D\tau_1)^{1/2}\\ &g \equiv g(0,\nu) \sim q_2(s,\nu)h(0,\nu,0,0) = (1+\nu\, v\tau_2)^{-1}\,(1-\nu^2 D\tau_1)^{-1}\\ &c = c(0,\nu) \sim h(0,0,0,\nu)q_2(0,\nu)\,h(0,\nu,\nu,0) = [1+\nu(D\tau_1)^{1/2}]^{-2}\,(1+\nu v\tau_2)^{-1}. \end{aligned} \tag{B.7}$$

Now, if we consider sequences of $2N$ regimes, beginning and finishing with regime 1, it is easily seen that we obtain $\sum_N \tilde{S}_{2N}(0)$ from (B.6) by changing τ_2 into τ_1, and we have

$$\langle T\rangle = (\tau_1+\tau_2)\frac{\frac{k}{1-pk}+\frac{qc}{(1-pk)(1-qg)}}{1-\frac{pqc}{(1-pk)(1-qg)}}. \tag{B.8}$$

We check that if $v\tau_2 \gg (D\tau_1)^{1/2}$, we have $p\sim 1$, $q \sim 0$ and we recover the approximation of large ballistic displacements

$$\langle T\rangle = (\tau_1+\tau_2)\frac{k}{1-k},$$

whereas for $v\tau_2 \ll (D\tau_1)^{1/2}$, we find the approximation of small ballistic displacements, equivalent to (A.7) in this limit:

$$\langle T\rangle = (\tau_1+\tau_2)\left(k+\frac{c}{1-g}\right).$$

Furthermore, if the target density ν vanishes, we have $k = g = c = 1$ and $\langle T\rangle \to \infty$, as it should be.

B.2. Limit behaviour of the mean search time

Considering as previously the case of low target density ν, we assume that $\varepsilon \equiv \nu\,(D\tau_1)^{1/2} \ll 1$ and write

$$\theta = v\tau_2/(D\tau_1)^{1/2}. \tag{B.9}$$

Then we have from (B.8)

$$\frac{\langle T\rangle}{\tau_1+\tau_2} \equiv \langle T\rangle = \frac{k-q(kg-c)}{1-pk-qg+pq(kg-c)}. \tag{B.10}$$

Using expressions (B.7) for k, g and c we find $kg - c = O(\varepsilon^2)$ and if $\varepsilon\theta \equiv \nu v\tau_2 \ll 1$

$$\langle T\rangle \sim \frac{1}{\varepsilon(2p+\theta q)}.$$

Noticing that by (B.2) $p = 1 - q = \theta^2(1+\theta)^{-2}$, we have

$$\langle T\rangle \sim \frac{\tau_1+\tau_2}{\nu(D\tau_1)^{1/2}[(2-\theta)p+\theta]} = \frac{\tau_1+\tau_2}{\nu v\tau_2}\frac{(1+\theta)^2}{1+4\theta}, \tag{B.11}$$

which correctly yields the limit behaviour (32) when $\theta \to 0$ and consequently $p \to 0$

$$\langle T\rangle \sim (\tau_1+\tau_2)\frac{1}{\nu v\tau_2} \qquad \text{if} \quad \nu(D\tau_1)^{1/2} \ll 1 \quad \text{and} \quad v\tau_2 \ll (D\tau_1)^{1/2}. \tag{B.12}$$

If $\theta \gg 1$, it can happen that $\varepsilon\theta \equiv \nu\, v\tau_2$ is not negligible in the limit of a small target density, $\varepsilon \ll 1$. Then (B.11) should be replaced by the more general formula

$$\langle T\rangle \sim \frac{\tau_1+\tau_2}{\nu\, v\tau_2}\frac{(1+\theta)^2(1+\varepsilon\theta)}{1+4\theta+2\varepsilon\theta^2} \sim \frac{\tau_1+\tau_2}{2\nu(D\tau_1)^{1/2}}\frac{1+\nu v\tau_2}{2+\nu v\tau_2}, \tag{B.13}$$

which yields the limit (B.12) when $\theta \to 0$, and the limit (28) when $\theta \to \infty$. This last limit is practically attained if $\nu v\tau_2$ is of the order of 8 or larger. In fact, it appears from numerical simulations that it is attained even for much smaller values of $\nu v\tau_2$, which indicates that p could be underestimated by formula (B.2) when $v\tau_2 \gg (D\tau_1)^{1/2}$.

B.3. Variations of $\langle T\rangle$ with τ_2

We study the possible minimum of $\langle T\rangle$, which surely does not occur for $v\tau_2 \gg (D\tau_1)^{1/2}$, where $\langle T\rangle$ is an increasing function of τ_2. Thus we assume that $\varepsilon\theta \equiv \nu\, v\tau_2 \ll 1$, so that we can use formula (B.11). Let us study the variations of $\langle T\rangle$ when τ_2 varies whereas τ_1 is kept constant. We have

$$\frac{1}{\langle T\rangle}\frac{\partial\langle T\rangle}{\partial\tau_2} = -\frac{\tau_1}{(\tau_1+\tau_2)\tau_2} + \left(\frac{2}{1+\theta} - \frac{4}{1+4\theta}\right)\frac{v}{(D\tau_1)^{1/2}}. \tag{B.14}$$

It is seen that the derivative $\partial\langle T\rangle/\partial\tau_2$ is negative if $\tau_2 \to 0$, positive if $\tau_2 \to \infty$, and it vanishes if

$$\left(\frac{\tau_1}{\tau}\right)^{1/2}\frac{1}{\theta} = \frac{4\theta^2-2\theta}{7\theta+1}, \tag{B.15}$$

where we used the characteristic time $\tau = D/v^2$. The left-hand side of (B.15) decreases to 0 with θ, whereas the right-hand side is negative for $0 < \theta < 1/2$, vanishes for $\theta = 1/2$, and it increases from 0 to ∞ when θ increases from $\frac{1}{2}$ to ∞. Thus $\langle T\rangle$ has one minimum for $\theta > 1/2$. Using the adimensional variables $\underline{\tau}_1 = \tau_1/\tau$, $\underline{\tau}_2 = \tau_2/\tau$ and noticing that $\theta = \underline{\tau}_2(\underline{\tau}_1)^{-1/2}$, we see that

- If $\underline{\tau}_1 \ll 1$, the solution θ_0 of (B.15) tends to $\frac{1}{2}$, and

$$\tau_2 \sim \tfrac{1}{2}(\tau_1)^{1/2}. \tag{B.16}$$

In this case, $\underline{\tau}_2 \gg \underline{\tau}_1$ and the corresponding value of $\langle T\rangle$ is

$$\langle T\rangle_{\min} \sim \frac{3}{4}\frac{L}{v}. \tag{B.17}$$

- If $\underline{\tau}_1 \gg 1$, the solution θ_0 of (B.15) satisfies $(\underline{\tau}_1)^{1/2}/\theta_0 \sim 4\theta_0/7$ and

$$\tau_2 \sim (7/4)^{1/2}(\underline{\tau}_1)^{3/4}. \tag{B.18}$$

In this case $\underline{\tau}_2 \ll \underline{\tau}_1$, $\theta_0 \gg 1$, $q = 1 - p \sim 2/\theta_0$, and $\langle T\rangle$ is found to be

$$\langle T\rangle_{\min} \sim \frac{1}{2}\frac{L}{v}\left(\frac{\tau_1}{\tau}\right)^{1/2}. \tag{B.19}$$

- If, eventually, we take $\theta = 2$, which is the value for which the approximations of large and small ballistic displacements coincide ($\theta = 2$ is the optimal value θ_0 if $\tau_1/\tau = (8/5)^2$), we recover the value of $\langle T\rangle$ given by (33).

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